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THE CALCULATION OF AIRCRAFT COLLISION PROBABILITIES

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16. Abstract

The basic limitation of air traffic compression, from the safety point of view, is the increased risk of collision due to reduced separations. In order to evolve new procedures, and eventually a fully automatic system, it is desirable to have a means of calculating the collision probability for any prescribed flight paths. This paper extends the statistical-probabilistic method of collision probability calculation, which has been limited to parallel, straight line flight paths, to arbitrary flight paths and vehicle shapes. The general formula is specialized to the cases of large relative velocity, non-zero relative velocity, zero relative velocity, and spherical collision surface. The formulas are applied to independent curved landing approaches to parallel runways.

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INTRODUCTION

THE PRIMARY responsibility of the air traffic controller is to insure the safety of the vehicles under his control. He does this, usually, by providing adequate separation at all times. What is *adequate separation*? The answer to this question is coming under increased scrutiny as the number of vehicles in the air increases and as efforts commence to automate the controller's task. In terminal areas, particularly, techniques for increasing airport capacity, such as closely-spaced parallel runways, curved approaches, speed control, and V/STOL operation, are limited, ultimately, by the closeness with which vehicles are allowed to approach each other. Flight paths must be such as to make most efficient use of the limited airspace available without incurring an unacceptable risk of collision. Very often the actual or proposed flight paths of two vehicles are such that the relative separation vector follows a complicated curvilinear path with no well-defined point of closest approach. A systematic design of new procedures, then, requires a quantitative relation between any prescribed set of flight paths and the risk of collision.

There are at least two ways to go about assessing collision risk between aircraft in motion: worst case analysis and statistical analysis.

A typical worst case analysis[1] combines initial position and velocity uncertainties, pilot control uncertainties, and possible changes in flight path due to pilot or controller intervention to determine the limits of the airspace within which the vehicle may be located in the time interval of interest. A conflict of volumes for two aircraft indicates that a collision may occur; and a warning is issued. This type of analysis is particularly well suited to a real-time collision warning system. Since it makes no assumptions about pilot or controller intentions, it can handle VFR as well as IFR traffic. But because great latitude must be allowed for the intentions of VFR aircraft, the volumes calculated expand rapidly in time. This limits the period of applicability to the order of seconds of time. Therefore it is usually impractical to use worst-case analysis to design most terminal or enroute traffic patterns; its major utility is in detecting and resolving potential conflicts in real time.

The statistical analysis of collision risk assumes that the pilot will proceed approximately along the path for which he has obtained clearance from the controller, and that the controller will not direct sudden, unsafe maneuvers. For these reasons, the method applies well to IFR traffic over a period of several minutes or even hours. The method has two aspects: (a) the collection and analysis of data to determine the statistics of

vehicle position and velocity deviations from the prescribed path, and (b) analysis of the distributions thus obtained to give the probability of collision. The first aspect requires combining equipment performance estimates with the results of pilot-vehicle simulations[2]. Very often the only information obtained about the distributions are their variances. As a result, the second aspect, which this paper treats, usually employs probability distributions based on analytical convenience rather than fact. Nevertheless, a series of analyses [3,4,5,6], based on assumed distributions, have developed risk estimates for parallel airways and, recently, for parallel runways[7]. A large number of cases, however, cannot be handled with present theory. Present theory deals directly with aircraft flying parallel, straight-line paths, and cases reducible to that. It cannot handle curved relative paths, ascending or descending airways, or crossing airways, although such cases are very common in the critical terminal areas in which most of the present congestion occurs. It is the purpose of this paper to provide the theoretical basis for calculating collision probabilities from the statistics for any two given paths in space. It is hoped that, as data are gathered for better definition of the distributions, the formulas here presented will assist in the development of improved terminal area traffic procedures and rules.

STATEMENT OF PROBLEM AND OF ASSUMPTIONS

Assume that two aircraft start on prescribed flight paths at time t_0. Due to navigation and control uncertainties the actual paths deviate from the prescribed paths. The question is: what is the probability of collision from time t_0 to some later time, t_1?

The following assumptions are made:

(1) All possible flight paths are random samples from ensembles of known statistics; the means of the ensembles are the prescribed paths.

(2) Each vehicle may be represented as occupying a bounded, closed region of space.

(3) Vehicle position is a differentiable function of time.

(4) The mean separation is always large compared to the diameter of the regions occupied by the vehicles.

The first assumption is the basis of the statistical-probabilistic analysis of aircraft collision hazards; its advantages and disadvantages already have been discussed.

The second assumption is needed to define a collision mathematically. But it is more than a mathematical nicety, for by appropriate selection of the vehicle volumes one may take account of wake turbulence, or calculate near-misses or air-space conflicts instead of actual collisions.

The third assumption is justified by the finite acceleration of aircraft.

The fourth assumption distinguishes collision analysis from interception analysis. It is well justified in practice since, by assumption (1), the mean separation is determined by the pre-planned flight paths, and these, it may be assumed, allow a separation that is large compared to vehicle dimensions.

ANALYSIS

Probability of Collision

It is desired to find the probability of at least one collision between the two vehicles in the interval $t_1 - t_0$. (More than one collision, although practically impossible, is not explicitly excluded by the assumptions. To do so would make the use of probability distributions extremely complicated.) Let N be the number of collisions between the two vehicles in time $t_1 - t_0$. The desired probability is $P[N \geq 1]$:

$$P[N \geq 1] = P[N = 1] + P[N = 2] + P[N = 3] + \cdots .$$

The terms $P[(N = 1) \cap (N = 2)]$, etc., do not enter because N cannot have two values at once.

Instead of calculating $P[N \geq 1]$ directly, it is easier to approximate it by $\tilde{N}$, the average of N taken over a large number of trials which are identical except for statistical fluctuations in the two paths. (The tilde (~) above a quantity indicates the ensemble average of the quantity; a vector is indicated by a bar – above the corresponding scalar magnitude. The terms *mean, average, expected* are used interchangeably in this paper to refer to the expectation value of a random variable.) By definition, $\tilde{N}$ is the mean of the probability distribution $P[N = i]$. That is, $\tilde{N} = 0 \cdot P[N = 0] + 1 \cdot P[N = 1] + 2 \cdot P[N = 2] + 3 \cdot P[N = 3] + \cdots$. The approximation to be used is $P[N \geq 1] \sim \tilde{N}$. The error in the approximation is, with $P[i]$ written for $P[N = i]$, $i = 1,2,3, \cdots$,

$$\tilde{N} - P[N \geq 1] = \sum_{i=1}^{\infty} i\ P[i] - \sum_{i=1}^{\infty} P[i] = \sum_{i=1}^{\infty} (i - 1)\ P[i]$$

$$= \sum_{1}^{\infty} i\ P[i + 1] \leq \sum_{1}^{\infty} i\ P[i]a$$

where $a \geq P[i + 1]/P[i]$

$$\tilde{N} - P[N \geq 1] \leq \sum_{1}^{\infty} i\ P[1]a^{i}$$

$$\leq \sum_{1}^{\infty} i\ P[N \geq 1]a^{i}, \text{ since } P[1] = P[N=1] \leq P[N \geq 1].$$

The percent error in the approximation is

$$(\tilde{N} - P[N \geq 1])/P[N \geq 1] \leq \sum_{i=1}^{\infty} ia^{i} = a/(1 - a)^{2},$$

assuming, $a < 1$.

The percent error is small provided only that a is small. The ratio a is just an upper bound on the probability of N+1 collisions, divided by the probability of N collisions. The condition on a is assured by qualifying assumption (1) as follows: The ensemble statistics are such that the probability of N collisions between the two vehicles in time $t_1 - t_0$ is large compared to the probability of N+1 collisions, for N>0.

The problem, then is to calculate $\tilde{N}$, the average number of collisions. In order to do this, it is necessary to make precise the notion of *collision*.

Description of "Collision Surface"

By assumption (2), each of the two vehicles may be represented as occupying a bounded, closed region of space, say V_1 and V_2. The center of volume of V_1 is defined as usual to be $\bar{r}_1$, $(= \int_{V_1} \bar{\rho}_1 dV_1 \,/\, \int_{V_1} dV_1)$ and so also for V_2 and $\bar{r}_2$. The position vectors $\bar{\rho}_1$ and $\bar{\rho}_2$ must have a common origin.

The *collision surface*, S_c, is now defined as the locus of $\bar{r}_2$ obtained by translating V_2 such that V_2 and V_1 have one or more common boundary points but no common points that are not boundary points. In other words, the collision surface is the center of the second vehicle when translated without rotation to touch but not penetrate the first. A two dimensional picture of a simple collision surface is given in Fig. 1. Note that the orientations of V_1 and V_2, and therefore the shape of S_c, may vary in time. Also, S_c is the boundary of a region V_c, which contains $\bar{r}_1$. Finally, it should be noted that the collision surface obtained by translating V_1 to touch V_2 is the reflection of S_c about the point $\bar{r}_2$. The subsequent analysis and the collision probabilities obtained from it are insensitive to which collision surface is used.

A *collision* now may be said to occur whenever the relative position vector $\bar{r}$ $(\equiv \bar{r}_2 - \bar{r}_1)$ enters S_c. However, this is not completely satisfactory if S_c has concavities, as shown in Fig. 2. For if the surface has concavities, the trajectory may enter S_c more than once in what should be counted as a single collision. There are at least three ways to avoid the

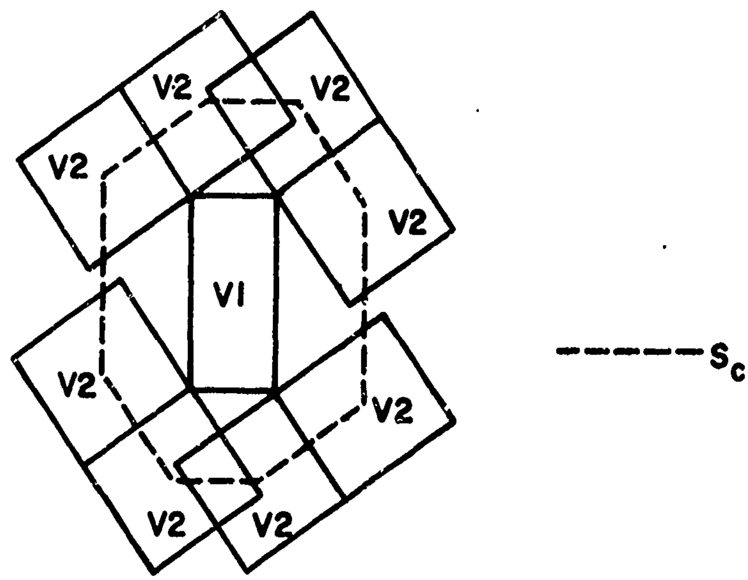

Figure 1. Collision Surface In Vehicles V_1 and V_2.

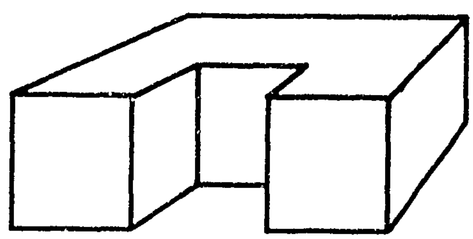

Figure 2. Collision Surface Typical of Trailing Vortices.

problem: (a) use a convex surface in place of one with concavities, (b) use a surface with concavities, but allow for the over-estimate in collision probability that will thus result, (c) use the *collision cross section vector*, to be defined subsequently. The first of these alternatives is preferred, since it greatly simplifies the computation. But only the third is rigorously correct.

Average Number of Collisions in $t_1 - t_0$

It is now possible to calculate $\tilde{N}$, the average number of collisions in time $t_1 - t_0$. The approach to be taken, following Rice[8], is to construct an expression for dN/dt, calculate its statistical mean, $\widetilde{dN/dt}$, and then integrate in time to get $\tilde{N}$. The mathematical legitimacy of the approach need not be discussed here.

An expression for dN/dt may be built upon the previous description of *collision*: a collision occurs whenever the relative position vector $\bar{r}$ enters the collision surface S_c. At such time, N increases abruptly from zero to unity. Let $\phi(\bar{r})$ be a step function in $\bar{r}$ space that is unity for $\bar{r}$ in V_c and zero for $\bar{r}$ outside V_c.

Then $\dot{N}$ may be defined:

$$\dot{N}(t) \equiv u(\bar{v}\cdot\bar{n})\ d\phi(\bar{r})/dt$$

where $\bar{v}$ is the relative velocity ($=d\bar{r}/dt$), $\bar{n}$ is the inward normal to S_c at point of entry and u(x) is zero for x < 0 and

unity for $x \geq 0$. The purpose of $u(\bar{v}\cdot\bar{n})$ in this expression is to exclude exits, for which $\bar{v}\cdot\bar{n} < 0$, but to include entries ($\bar{v}\cdot\bar{n} > 0$) and mere touches ($\bar{v}\cdot\bar{n} = 0$).

The time derivative $\dot{\phi}(\bar{r})$ may be written as $d\phi/dr$, which is a delta function, times dr/dt provided the position, $\bar{r}$, is a differentiable function of time. Since this is so, by assumption (3), it follows that

$$\dot{N} = u(\bar{v}\cdot\bar{n})\,(d\phi/dr)(dr/dt) = u(\bar{v}\cdot\bar{n})\bar{v}\cdot\nabla\phi.$$

To obtain the ensemble average $\tilde{\dot{N}}$, it is necessary to consider, instead of the single relative trajectory, an ensemble of trajectories. At any instant, the probability density distribution $W(\bar{r},\bar{v},t)$ of the relative position and velocity is known, by assumption. Therefore, the ensemble average may be found as

$$\tilde{\dot{N}} = \int d(\bar{v}) \int d(\bar{r})\, W(\bar{r},\bar{v},t)\; u(\bar{v}\cdot\bar{n})\; \bar{v}\cdot\nabla\phi.$$

Here $d(\bar{v})$ and $d(\bar{r})$ are elements of velocity space and position space. The integrations are understood to cover the spaces completely. By utilizing the identity $\nabla\cdot(\phi\bar{x}) = \bar{x}\cdot\nabla\phi + \phi\nabla\cdot\bar{x}$, where ϕ is a scalar and $\bar{x}$ a vector, say $\bar{x} = \bar{v}\,u(\bar{v}\cdot\bar{n})\,W(\bar{r},\bar{v},t)$, the above is written

$$\begin{aligned}\tilde{\dot{N}} &= \int d(\bar{v}) \int d(\bar{r})\; \bar{x}\cdot\nabla\phi \\ &= \int d(\bar{v}) \int d(\bar{r}) \left[\nabla\cdot(\phi\bar{x}) - \phi\,\nabla\cdot\bar{x}\right] \\ &= \int d(\bar{v})\left[\int d(\bar{r})\;\nabla\cdot(\phi\bar{x}) - \int d(\bar{r})\;\phi\;\nabla\cdot\bar{x}\right].\end{aligned}$$

By the divergence theorem the first integral in the braces, which is over the volume of $\bar{r}$ space, is converted to a surface integral over a sphere S_R of increasingly large radius R, centered at $\bar{r} = 0$:

$$\int d(\bar{r})\ \nabla\cdot(\phi\bar{x}) = \lim_{R\to\infty} \int_{S_R} d\bar{s}\cdot(\phi\bar{x}).$$

As $R\to\infty$ the surface S_R eventually lies wholly outside of S_c. Since $\phi = 0$ outside S_c, the limiting value is zero.

The second volume integral in the brackets is zero in the space outside S_c, because ϕ is zero there. It has a non-zero value only within the collision volume V_c, where $\phi = 1$. Therefore

$$\dot{\tilde{N}} = -\int d(\bar{v}) \int_{V_c} d(\bar{r})\,(1)\ \nabla\cdot\bar{x}$$

$$= \int d(\bar{v}) \int_{S_c} d\bar{s}\cdot\bar{x}, \qquad \text{where } d\bar{s} = (ds)\bar{n}$$

$$= \int d(\bar{v}) \int_{S_c} (d\bar{s}\cdot\bar{v})\ u(\bar{v}\cdot\bar{n})\ W(\bar{r}_c,\bar{v},t)$$

$$= \int d(\bar{v})\ W(\bar{0},\bar{v},t)\,\bar{v}\cdot \left[\int_{S_c} d\bar{s}\ u(\bar{v}\cdot\bar{n})\right].$$

$$\tilde{N} = \int_{t_0}^{t_1} \dot{\tilde{N}}\, dt\ .$$

Two notes regarding the above are in order.

First, $\bar{r}_c$, the relative position vector drawn to the collision surface S_c has been replaced by $\bar{0}$, the null vector in $\bar{r}$ space. The reason is that the distribution W in $\bar{r}$ will have a mean large compared to $\bar{r}_c$ in all practical cases, as stated in assumption (4). The difference in the density W for $\bar{r} = \bar{r}_c$ and $\bar{r} = \bar{0}$, therefore, is small. As a result, W may be taken outside the surface integral.

Second, the surface integral $\int_{S_c} d\bar{s}\ u(\bar{v}\cdot\bar{n})$ may be interpreted as a *collision cross-section vector* with respect to $\bar{v}$, and written $\overline{\Delta S}_c(\bar{v})$. If S_c has no concavities, the vector $\overline{\Delta S}_c$ has the property that its component in the $\bar{v}$ direction is the area of the orthogonal projection of S_c onto a plane perpendicular to $\bar{v}$. If S_c does have concavities, then this property, rather than the surface integral, serves to define $\overline{\Delta S}_c$. In either case one has

$$\tilde{N} = \int_{t_0}^{t_1} dt \left[\int d(\bar{v})\ W(\bar{0},\bar{v},t)\ \bar{v}\cdot\overline{\Delta S}_c(\bar{v}) \right] . \tag{1}$$

The general formula thus derived is a three-dimensional form of the well-known expression for the mean number of times a random function y(t) crosses a given level y_o in a time interval $t_1 - t_0$. The one-dimensional formula, originally derived by S.O. Rice[8], is

$$\int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} W_1(y_o,\dot{y},t)\ |\dot{y}|\ d\dot{y}$$

where w_1 is the joint probability density of y and $\dot{y}$. The three-dimensional formula gives the average number of times a random vector $\bar{r}(t)$ enters a given closed surface S_c in time $t_1 - t_0$. The major difference between the two formulas is that the one-dimensional case requires specification of a level, y_0, while the three-dimensional case requires specification of an entire surface, S_c. The term $|\dot{y}|$ of the former case becomes the more complicated integral $\bar{v}\cdot\overline{\Delta S}_c$:

$$\bar{v}\cdot\overline{\Delta S}_c = \bar{v}\cdot \int_{S_c} u(\bar{v}\cdot\bar{n})\ \bar{n}\ ds$$

Special Case: Large Relative Velocity

The simplest case, mathematically, occurs when the mean relative velocity is large compared to the spread in the distribution of velocity. Fortunately this case covers many practical situations, such as curved landing approaches, crossing airways, climbing and holding patterns and overflights of landings or takeoffs. The situation may be approximated by assuming the density $W(\bar{0},\bar{v},t)$ to be a delta function in velocity space at the mean velocity $\tilde{\bar{v}}$. In that case,

$$\int d(\bar{v})\ \bar{v}\ W(\bar{0},\bar{v},t) = \tilde{\bar{v}}(t)\ W_r(\bar{0},t)$$

where $W_r(\bar{r},t)$ is the density distribution in $\bar{r}$. As a result, $\tilde{N}$ becomes

$$\tilde{N} = \int_{t_0}^{t_1} dt\ \tilde{\bar{v}}\ (t) \cdot \overline{\Delta S}_c\ W_r(\bar{0},t)$$

$$= \int_{t_0}^{t_1} dt\ (d\tilde{\bar{r}}/dt) \cdot \overline{\Delta S}_c\ W_r(\bar{0},t)$$

$$= \int_{\tilde{\bar{r}}(t_0)}^{\tilde{\bar{r}}(t_1)} d\tilde{\bar{r}} \cdot \overline{\Delta S}_c\ W_r'(\tilde{\bar{r}},t), \tag{2}$$

where W_r' is W_r with mean translated to $\bar{0}$.

In this formula, $\tilde{N}$ may be visualized as the volume swept out through the position distribution W_r by the collision surface S_c, in going from $\tilde{\bar{r}}(t_o)$ to $\tilde{\bar{r}}(t_1)$. Such a picture is useful if the relative path is irregular or if the density distribution is time-dependent. In such cases it is practical to program the integration of (2) on a digital computer using relatively large step sizes and simple integration algorithms. An order of magnitude estimate of $\tilde{N}$ is usually adequate.

If the relative path is a straight line and if the density distribution is time-independent and analytically convenient, further analysis of equation (2) can be carried out. The following analysis assumes straight line paths and time-invariant Gaussian statistics, a case treated *ad hoc* in the literature[1,3,4].

Let the vehicles be represented by two right circular cylinders with parallel bases of diameters d_1 and d_2 and heights h_1 and h_2. Let the mean paths $\tilde{\bar{r}}_1(t)$ and $\tilde{\bar{r}}_2(t)$ be horizontal straight lines going to infinity in both directions;

finally, let the actual positions $\bar{r}_1$ and $\bar{r}_2$ be normally and independently distributed with stationary statistics about the mean positions:

$$W_{r1}(\bar{r}_1) = (2\pi)^{-3/2} \, |K_{r1}|^{-1/2} \exp - \frac{1}{2} \left[(\bar{r}_1-\tilde{\bar{r}}_1)_T K_{r1}^{-1} (\bar{r}_1-\tilde{\bar{r}}_1) \right] ,$$

$$W_{r2}(\bar{r}_2) = (2\pi)^{-3/2} \, |K_{r2}|^{-1/2} \exp - \frac{1}{2} \left[(\bar{r}_2-\tilde{\bar{r}}_2)_T K_{r2}^{-1} (\bar{r}_2-\tilde{\bar{r}}_2) \right] .$$

Here subscript T denotes transpose and K_{r1}^{-1} and K_{r2}^{-1} are the inverses of the covariance matrices K_{r1} and K_{r2}. These covariances are assumed to be known in the $(x_1y_1z_1)$ and $(x_2y_2z_2)$ coordinate systems illustrated in Fig. 3. In the figure, the position errors $(\bar{r}_1 - \tilde{\bar{r}}_1)$ and $(\bar{r}_2 - \tilde{\bar{r}}_2)$ have components x_1 and x_2 in the direction of travel, z_1 and z_2 along the vertical and y_1 and y_2 normal to the other two components. In these coordinates, assuming the position errors are statistically independent in the three directions, the covariances are

$$K_{r1} = \begin{bmatrix} \sigma_{1x}^2 & 0 & 0 \\ 0 & \sigma_{1y}^2 & 0 \\ 0 & 0 & \sigma_{1z}^2 \end{bmatrix} \qquad K_{r2} = \begin{bmatrix} \sigma_{2x}^2 & 0 & 0 \\ 0 & \sigma_{2y}^2 & 0 \\ 0 & 0 & \sigma_{2z}^2 \end{bmatrix}$$

To calculate $\tilde{N}$ in this case, it is necessary first to compute the distribution of $\bar{r}_2 - \bar{r}_1$ and then to compute the portion of that distribution swept out by a right vertical circular cylinder, height $h_1 + h_2$ and diameter $d_1 + d_2$, moving on the straight line $\tilde{\bar{r}}_2 - \tilde{\bar{r}}_1$.

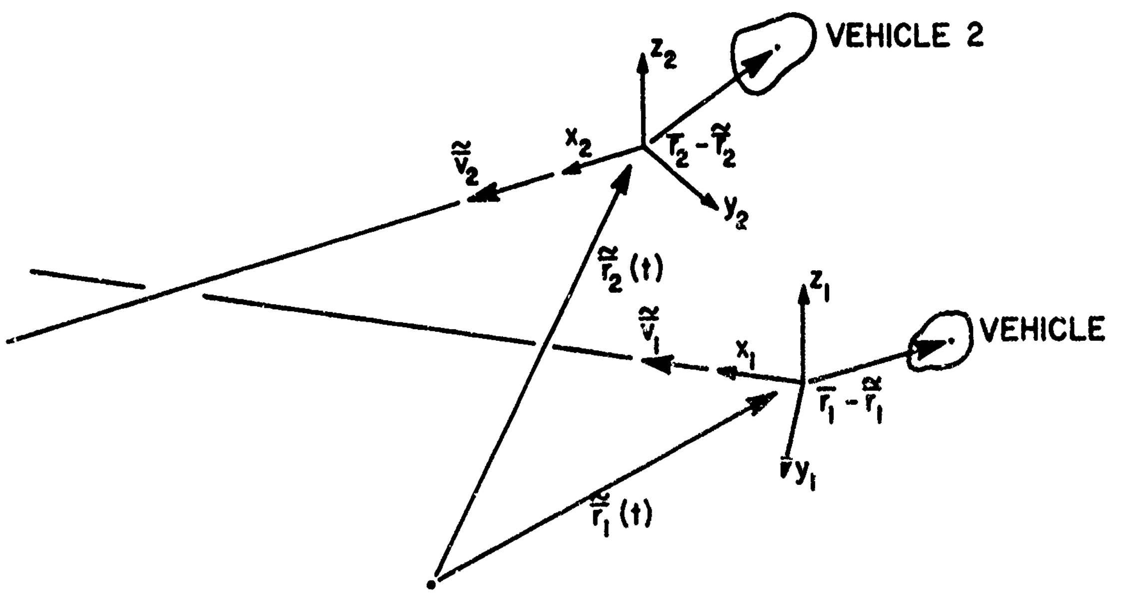

Figure 3. Geometry of Straight-line Paths.

The distribution of $\bar{r}_2 - \bar{r}_1$ is normal, being the sum of normal random variables. If the position errors of the two vehicles are independent, as assumed for this example, then

$$K_r = K_{r2} + K_{r1} = \begin{bmatrix} \sigma_x{}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{yx}^2 & \sigma_y{}^2 & \sigma_{yz}^2 \\ \sigma_{zx}^2 & \sigma_{zy}^2 & \sigma_z{}^2 \end{bmatrix} .$$

The σ^2 terms here are determined by writing K_{r1} and K_{r2} in the same (x y z) coordinate system, which is yet to be selected, and adding. The distribution $W_r(\bar{0})$ is

$$W_r(\bar{0}) = (2\pi)^{-3/2} \, |K_r|^{-1/2} \exp - \frac{1}{2} \left[(-\tilde{\bar{r}}) \, K_r^{-1} \, (-\tilde{\bar{r}})_T\right]$$
$$\equiv W_r'(\tilde{\bar{r}}).$$

The portion of this distribution swept out by the collision surface is given by formula (2). Choosing x to be along the relative velocity vector, y horizontal and perpendicular to x, and z upward, as shown in Fig. 4, formula (2) leads to:

$$N = (h_1+h_2)(d_1+d_2)/(2\pi \, \sigma_z \sigma_y) \exp - 1/2 \left((\tilde{y}^2/\sigma_y{}^2) + (\tilde{z}^2/\sigma_z{}^2) \right)$$

where

$$\sigma_x{}^2 \equiv \sigma_{1x}^2 \cos^2 a + \sigma_{2x}^2 \cos^2 \beta + \sigma_{1y}^2 \sin^2 a + \sigma_{2y}^2 \sin^2 \beta$$

$$\sigma_y{}^2 \equiv \sigma_{1y}^2 \cos^2 a + \sigma_{2y}^2 \cos^2 \beta + \sigma_{1x}^2 \sin^2 a + \sigma_{2x}^2 \sin^2 \beta$$

$$\sigma_{xy}^2 \equiv \frac{1}{2}(\sigma_{1y}^2 - \sigma_{1x}^2) \sin 2a + \frac{1}{2}(\sigma_{2y}^2 - \sigma_{2x}^2) \sin 2\beta$$

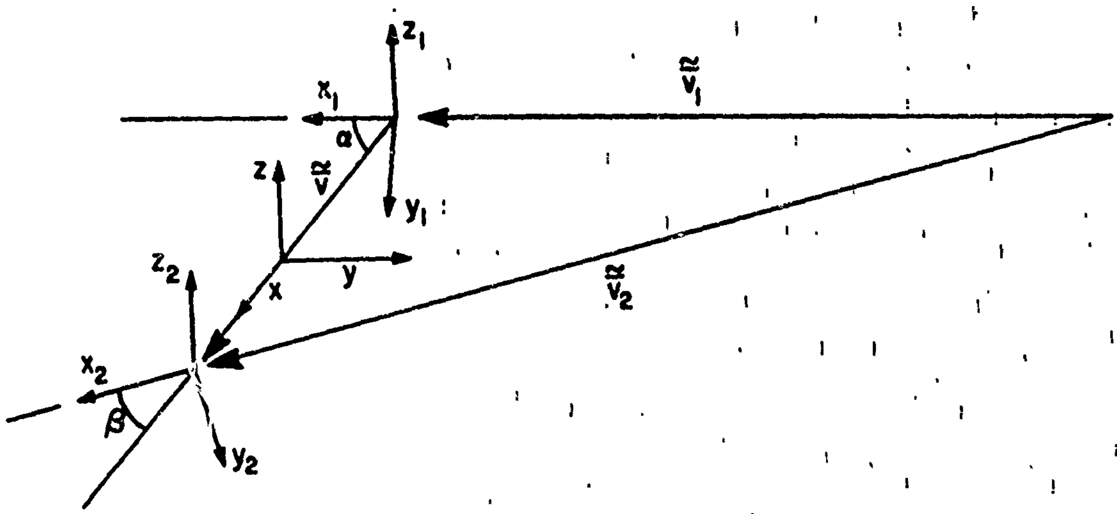

Figure 4. Relative Geometry of Straight-Line Paths.

$$\sigma_z'^2 \equiv \sigma_{1z}^2 + \sigma_{2z}^2$$

$$\sigma_{xz}^2 = \sigma_{yz}^2 = 0 \qquad\qquad \mu \equiv \sigma_{xy}^2 / \sigma_x\sigma_y \; .$$

For the special case of vehicles passing on the same level in opposite directions with mean path separation S_y,

$$\tilde{y} = S_y, \quad \alpha = \pi , \quad \sigma_y^2 = \sigma_{2y}^2 + \sigma_{1y}^2 ,$$

$$\tilde{z} = 0 , \quad \beta = 0 , \quad \sigma_z^2 = \sigma_{2z}'^2 + \sigma_{1z}^2 ,$$

$$\tilde{N} = \left((h_1+h_2)(d_1+d_2)/2\pi\right) / \left((\sigma_{1z}^2+\sigma_{2z}^2)^{1/2} \; (\sigma_{1y}^2+\sigma_{2y}^2)^{1/2}\right)$$

$$\exp - \frac{1}{2} \left[(S_y^2)/(\sigma_{2y}'^2+\sigma_{1y}^2) \right]$$

If, further, it is assumed that $h_1 = h_2$, $d_1 = d_2$, $\sigma_{1y}^2 = \sigma_{2y}^2$, and $\sigma_{1z}^2 = \sigma_{2z}^2$, then the value of $\tilde{N}$ coincides exactly with that obtained by Reich[3] and Taylor.[4]

Special Case: Non-Zero Relative Velocity

If the mean relative velocity is not large compared to the higher moments, the delta-function approximation for the velocity distribution cannot be made. This situation occurs when there is a small (but non-zero) relative velocity, as, for example, when two aircraft are controlled to be in train preparatory to final approach[2], or to be following parallel enroute airways but with slightly different speeds.

It will be assumed for the present case only that the mean relative velocity and the higher moments are all different from zero and that the position errors are independent of velocity errors. Then the joint distribution for each vehicle may be factored:

$$W_1(\bar{r}_1,\bar{v}_1,t) = W_{1r}(\bar{r}_1,t)\ W_{1v}(\bar{v}_1,t) \ ,$$

$$W_2(\bar{r}_2,\bar{v}_2,t) = W_{2r}(\bar{r}_2,t)\ W_{2v}(\bar{v}_2,t) \ .$$

Since relative position $\bar{r}$ and relative velocity $\bar{v}$ are sums of random variables, they are independent if $(\bar{r}_1+\bar{r}_2)$ is independent of $(\bar{v}_1+\bar{v}_2)$. If so, $W(\bar{r},\bar{v},t)$ also may be factored so that formula (1) is

$$\tilde{N} = \int_{t_0}^{t_1} dt \left[\int d(\bar{v})\ W_v(\bar{v},t)\ \bar{v}\cdot\overline{\Delta S}_c \right] W_r(\bar{0},t) \ . \tag{3}$$

If the collision cross-section and the velocity distribution are independent of time, the portion in square brackets may be taken outside the time integral. Nevertheless, a computer solution is usually necessary. One case that can be handled analytically is that of one aircraft slowly overtaking and passing another on a parallel airway, with normal statistics. The analysis[9] is tedious and hence only the result will be given here.

If the statistics are normal and if the vehicles are represented by vertical circular cylinders of heights h_1 and h_2, diameters d_1 and d_2, (see Figure 3) moving on horizontal straight paths, then the collision probability is approximately

$$(d_1+d_2)^2\lambda_z' + (d_1+d_2)(h_1+h_2)\sqrt{2}\sqrt{\lambda_x'^2+\lambda_y'^2}\ f(\mu,\kappa)/(4\sqrt{2\pi}\ \sigma_z\sigma_y\ |\tilde{v}|)$$

$$\cdot\ \exp - \frac{1}{2}\left[(\tilde{y}^2/\sigma_y^2) + (\tilde{z}^2/\sigma_z^2) \right] .$$

where

$$f(\mu,\kappa) \equiv e^{-\mu} \begin{bmatrix} (1+\mu)^2 \cdot J_o(i\mu) + \mu^2 J_2(i\mu) \\ -\frac{1}{2}\kappa^2 \cos 2\tilde{\phi} \left[\mu(\mu+\frac{3}{2})J_o(i\mu) + \mu(\mu-\frac{1}{2})J_2(i\mu)\right] \end{bmatrix}$$

$$\mu \equiv \frac{1}{4}\left[(\tilde{v}_x'^2/\lambda_x'^2)+(\tilde{v}_y'^2/\lambda_y'^2)\right]$$

$$\kappa^2 \equiv (\lambda'^2_y - \lambda'^2_x) \;/\; (\lambda'^2_x + \lambda'^2_y)$$

$J_n \equiv$ Bessel Function, order n, first kind; $i = \sqrt{-1}$

$$\tan\tilde{\phi} \equiv \tilde{v}_y' \lambda_x' / \tilde{v}_x' \lambda_y' = (\lambda_x' / \lambda_y') \tan\theta$$

$$\tilde{v}_x' \equiv \tilde{v}_2 \cos(\beta+\theta) - \tilde{v}_1 \cos(a+\theta) = \tilde{v}\cos\theta$$

$$\tilde{v}_y' \equiv \tilde{v}_2 \sin(\beta+\theta) - \tilde{v}_1 \sin(a+\theta) = \tilde{v}\sin\theta$$

$$\lambda'^2_x \equiv \frac{1}{2}(\lambda^2_{1x} + \lambda^2_{2x} + \lambda^2_{1y} + \lambda^2_{2y}) - \frac{1}{2}\Big((\lambda^2_{1y}-\lambda^2_{1x})^2 + (\lambda^2_{2y}-\lambda^2_{2x})^2 + 2(\lambda^2_{1y}-\lambda^2_{1x})(\lambda^2_{2y}-\lambda^2_{2x})\cos 2(a-\beta)\Big)^{1/2}$$

$$\lambda'^2_y \equiv \frac{1}{2}(\lambda^2_{1x} + \lambda^2_{2x} + \lambda^2_{1y} + \lambda^2_{2y}) + \frac{1}{2}\Big((\lambda^2_{1y}-\lambda^2_{1x})^2 + (\lambda^2_{2y}-\lambda^2_{2x})^2 + 2(\lambda^2_{1y}-\lambda^2_{1x})(\lambda^2_{2y}-\lambda^2_{2x})\cos 2(a-\beta)\Big)^{1/2}$$

$\sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$: as in the case of large relative velocity

$$\lambda'^2_z \equiv \lambda^2_{1z} + \lambda^2_{2z}$$

$$\tan 2\theta \equiv \left[(\lambda^2_{1y}-\lambda^2_{1x})\sin 2a + (\lambda^2_{2y}-\lambda^2_{2x})\sin 2\beta\right] \Big/ \left[(\lambda^2_{1y} - \lambda^2_{1x})\cos 2a + (\lambda^2_{2y}-\lambda^2_{2x})\cos 2\beta\right]$$

In the above, vehicle 1 has position variances σ_{1x}^2, σ_{1y}^2, σ_{1z}^2 and velocity variances λ_{1x}^2, λ_{1y}^2, λ_{1z}^2 which are along its track (x), across its track (y), and along the vertical (z); its mean velocity $\tilde{\bar{v}}_1$ makes an angle α with the mean relative velocity vector $\tilde{\bar{v}}$. (See Fig. 4) Analogous terms σ_{2x}^2, σ_{2y}^2, σ_{2z}^2, λ_{2x}^2, λ_{2y}^2, λ_{2z}^2, $\tilde{\bar{v}}_2$, β hold for vehicle 2. The mean vertical path separation is $\tilde{z}$, the closest horizontal approach of the mean paths is $\tilde{y}$. The six position and velocity errors of a vehicle have been assumed uncorrelated with each other and with those of the other vehicle.

Special Case: Zero Relative Velocity

When two aircraft using the same airway, or in train to the same landing approach are controlled so that (on the average) their separation is constant, then the mean relative velocity is zero. This case must be handled separately, because if in the preceding result, the mean relative velocity $\tilde{\bar{v}}$ is allowed to approach zero, the value of $\tilde{N}$ increases without bound. Physically, such a result is expected, since $\tilde{\bar{N}}$ is constant in time, and $\int_{-\infty}^{\infty} dt\, \tilde{\bar{N}}$ must be unbounded. The infinite limits of time should be replaced by the more realistic limits t_0 and t_1. If the statistics are stationary with velocity independent of position, the formula for $\tilde{N}$ is

$$\tilde{N} = \int_{t_0}^{t_1} dt \left[\int d(\bar{v})\; W(\bar{0},\bar{v})\bar{v}\cdot\overline{\Delta S}_c \right]$$

$$= (t_1 - t_0)\left[\int d(\bar{v})\; W_{\bar{v}}(\bar{v})\bar{v}\cdot\overline{\Delta S}_c \right] W_r(\bar{0}) \;. \qquad (4)$$

To illustrate an analytic solution in the case of zero relative velocity, the same geometry will be assumed as for the two preceding cases, i.e., two right cylinders moving on the mean straight horizontal paths shown in Fig. 3. (The statistics are assumed to be gaussian and stationary, with no correlation between vehicles.) The result obtained[9] is:

$$\frac{(t_1 - t_0)}{\sigma_x\sigma_y\sigma_z}\,\frac{\bar{\lambda}}{2}\left[1 - \sum_{n=0}^{\infty}\frac{1\cdot3\cdot5\cdots(4n-3)}{[2\cdot4\cdot6\cdots2n]^2}\left(\frac{\kappa^2}{2}\right)^{2n}\right]\exp - \frac{1}{2}\left[\frac{\tilde{x}^2}{\sigma_x^{\,2}}+\frac{\tilde{y}^2}{\sigma_y^{\,2}}+\frac{\tilde{z}^2}{\sigma_z^{\,2}}\right]$$

where

$$\bar{\lambda}^2 \equiv \frac{1}{2}\,(\lambda_{1x}^2 + \lambda_{2x}^2 + \lambda_{1y}^2 + \lambda_{2y}^2)$$

$$\kappa^2 \equiv \frac{1}{2}\,(\lambda_{1y}^2 + \lambda_{2y}^2 - \lambda_{1x}^2 - \lambda_{2x}^2)\;/\;\bar{\lambda}^2$$

$$\sigma_x^{\,2} = \sigma_{1x}^2 + \sigma_{2x}^2$$

$$\sigma_y^{\,2} = \sigma_{1y}^2 + \sigma_{2y}^2\,.$$

Special Case: Spherical Collision Surface

When the collision surface S_c is spherical the collision cross section vector $\Delta\bar{S}_c$ is $\hat{v}\pi a^2$ where $\hat{v}$ is a unit vector in the direction of $\bar{v}$ and $a(t)$ is the radius of the sphere at time t. Then the expression for $\tilde{N}$ is

$$\tilde{N} = \int_{t_0}^{t_1} dt \int d(v)\; W(\bar{o},\bar{v},t)\;\bar{v}\cdot\hat{v}\;\pi\, a^2(t)$$

$$= \int_{t_0}^{t_1} dt \quad W_r(\bar{o},t)\;\tilde{v}(t)\;\pi\, a^2(t) \qquad (5)$$

where $\tilde{v}(t)$ is a scalar, equal to the average magnitude of $d\bar{r}/dt$. This quantity, $\tilde{v}(t)$, is just the mean relative speed; it is never negative and is zero only when the velocity distribution has zero mean and no spread (a delta function at the origin). Unlike (2), formula (5) is not restricted to large relative velocity. Its only limitation is that it applies to spherical collision surfaces.

In many cases an actual collision surface may be replaced by a spherical one of equal area without seriously impairing the usefulness of the result. This occurs, for example, in determining the probability of near misses, or in obtaining an order-of-magnitude answer. If such an approximation may be made then the collision probability is given by (5) with A(t), one fourth the area of the actual collision surface at time t, in place of $\pi\alpha^2(t)$.

APPLICATION TO PARALLEL RUNWAYS

The use of closely spaced, independently operated, parallel runways is one means to increase airport capacity that was recommended for further investigation by the Air Traffic Advisory Committee Report.[10] The scheme involves multiple curved precision approaches to two or more parallel runways spaced at less than the present 5000-ft minimum. No time synchronism is assumed between aircraft on different tracks. The safety of such approaches depends on at least four factors: the probability of missed approaches,[7] emergency (e.g., engine out) procedures, accuracy of guidance of each aircraft relative to its runway, accuracy of guidance of each aircraft relative to the other. The last of these factors was analyzed in detail by the method developed in this paper, and that analysis will be described next.

A computer program was written to calculate the collision probability for two aircraft making the simultaneous curved approaches to parallel runways shown in Fig. 5. One aircraft was assumed to be relatively fast (approach speed 120 knots), the other relatively slow (approach speed 80 knots). The details of the approaches are given in Table 1 and Fig. 5. The aircraft volumes were taken to be right circular cylinders.

The statistics were assumed to be Gaussian and the standard deviations of position in the y and z directions were assumed to diminish as they approached the runway (see Fig. 6). This improved accuracy occurs, in practice, for

TABLE 1.- DETAILS OF CURVED APPROACHES

	Vehicle 1	Vehicle 2	Units
Path Segment 1			
duration	315.	60.	sec
ground speed	133.	200.	ft/sec
descent rate	10.	13.3	ft/sec
initial altitude	6600.	9200.	ft
Path Segment 2			
duration	45.	180.	sec
ground speed	133.	200.	ft/sec
descent rate	10.	13.3	ft/sec
initial altitude	3450.	8400.	ft
Path Segment 3			
duration	450.	600.	sec
ground speed	133.	200.	ft/sec
descent rate	6.7	10.	ft/sec
initial altitude	3000.	6000.	ft
Path Segment 4			
duration	90.	60.	sec
deceleration	1.48	3.33	ft/sec/sec
descent rate	0.	0.	ft/sec
initial altitude	0.	0.	ft

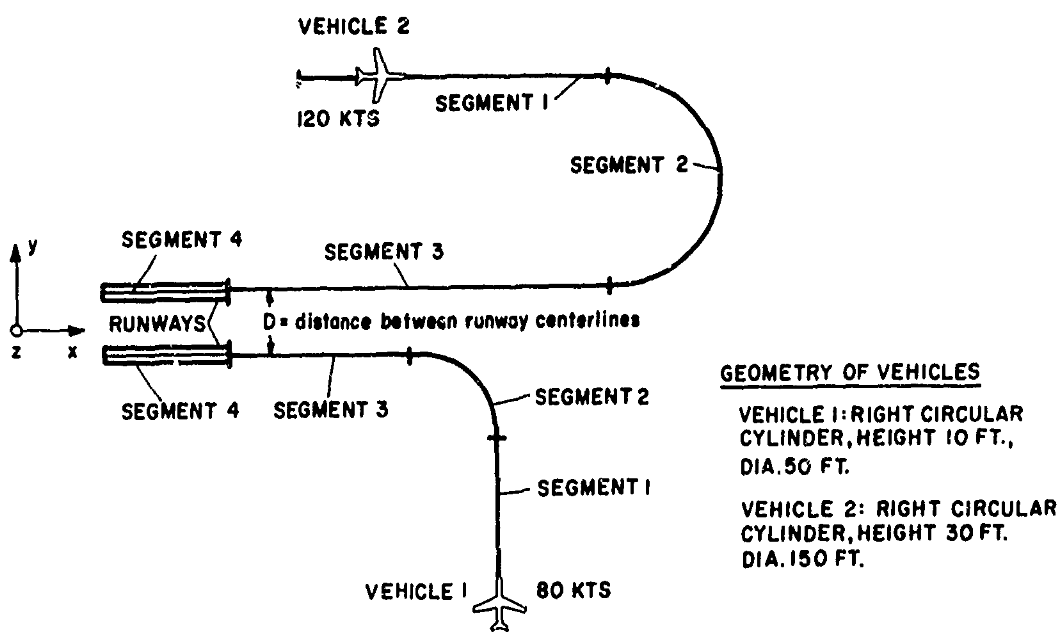

Figure 5. Geometry of Simultaneous Curved Approaches.

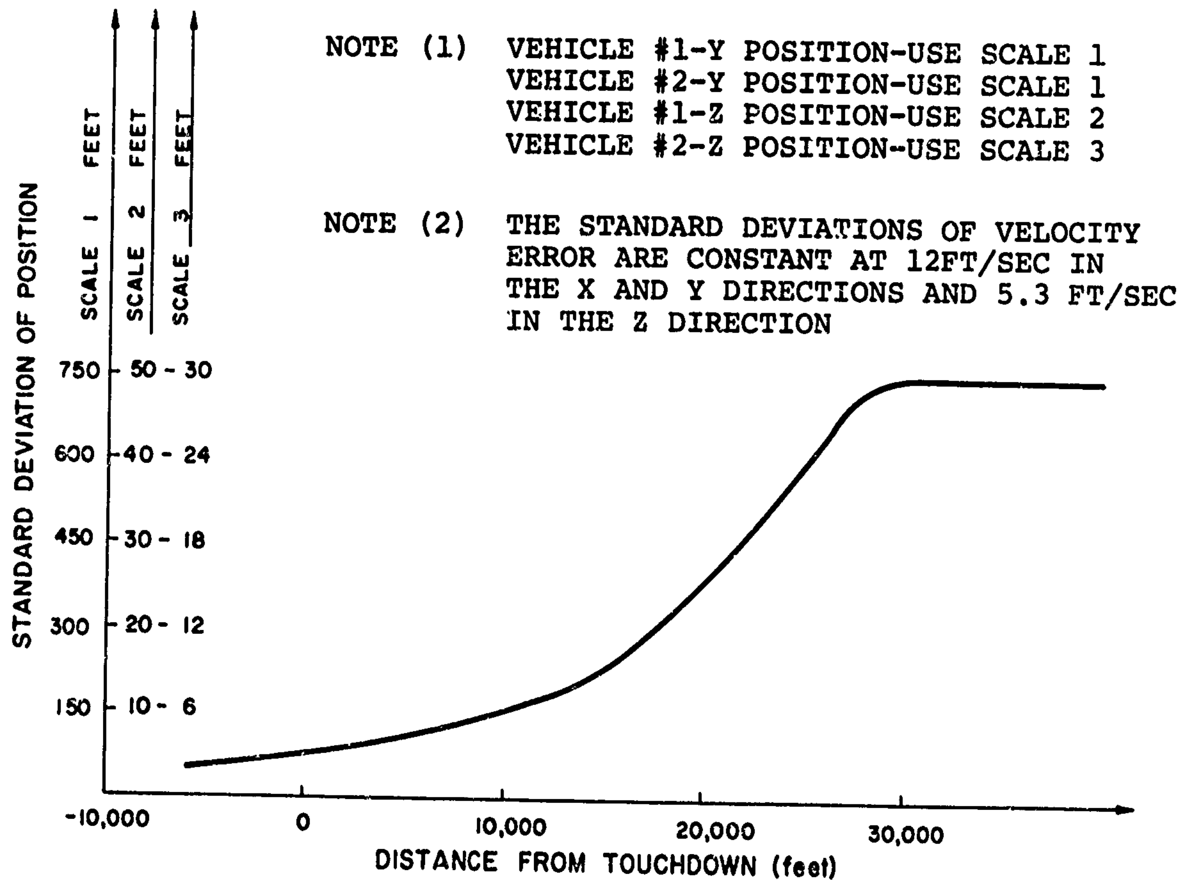

Figure 6. Standard Deviations of Position Vs. Distance from Touchdown.

several reasons: The ILS errors are approximately angular errors, and diminish as the source is approached. In addition, aircraft control errors diminish with length of time on the beam and, finally, visual cues improve as touchdown is approached. None of these circumstances, however, improves positional guidance accuracy in the direction of the runway, so the standard deviations of the errors in the x direction were assumed constant at 6000 ft for each vehicle.

The formula for the large relative velocity case given by (2) was found to be adequately accurate even though the relative velocity passed through zero shortly after the touchdowns. This accuracy was verified by comparing the result of using (2) with that obtained from the exact integration of (1), which does not contain any assumptions on relative velocity. The two answers agreed to within 8 percent. Since the computation time for the large relative velocity case (2) is about 1/20 that for the exact integration, formula (2) was used throughout the runs. It should be noted that formula (2) does not assume independence of position and velocity statistics and, in fact, requires no velocity statistics at all. Further, although Gaussian position statistics were assumed, any other statistics may have been used in the computer runs with little change in the program.

RESULTS OF COMPUTER RUNS

The results of the computer runs are shown in Figs. 7 through 9. The data of Figs. 5 and 6, and of Table 1 apply to all runs except as noted in the following discussions.

Fig. 7: (Variation of collision probability with time between touchdown.) If the two aircraft come abreast at the touchdown point (T=0) the collision probability is very small, less than 10^{-38}. As T increases, the slower aircraft (vehicle 1) touches down later than the faster aircraft (vehicle 2) and the point of passing occurs farther up the glide slope. Since the lateral position errors increase with distance from touchdown, (Fig. 6), the collision probability increases as the point of passage moves away from the touchdown point. The sharp drop in collision probability beyond T = 100 is due to the geometry of the turn-on for vehicle 1.

In order to obtain conservative estimates, the runs of Fig. 8 and Fig. 9 were made with T = 80, which maximizes the collision probability. If the runways were truly independently operated, the times between touchdowns would be distributed uniformly between 0 and the time, τ, between successive landings on the same runway. Since τ often is about 120 seconds, the assumption is not overly pessimistic.

Fig. 8: (Variation of collision probability with runway separation.) The Figure shows that cutting the runway separation from 5000 ft to 2500 ft increases the probability of a collision by a factor of 6000, other things being equal.

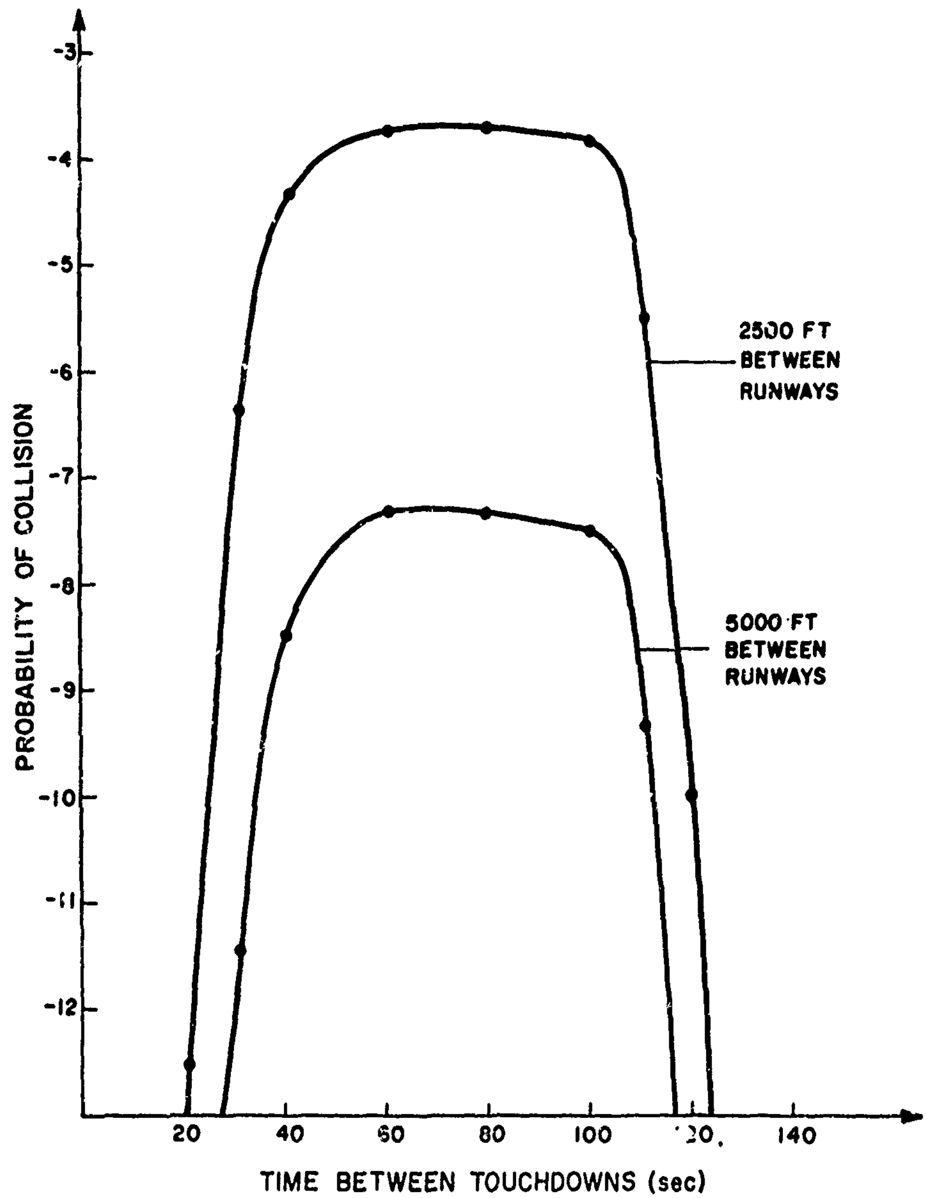

Figure 7. Probability of Collision Vs. Time Between Touchdowns

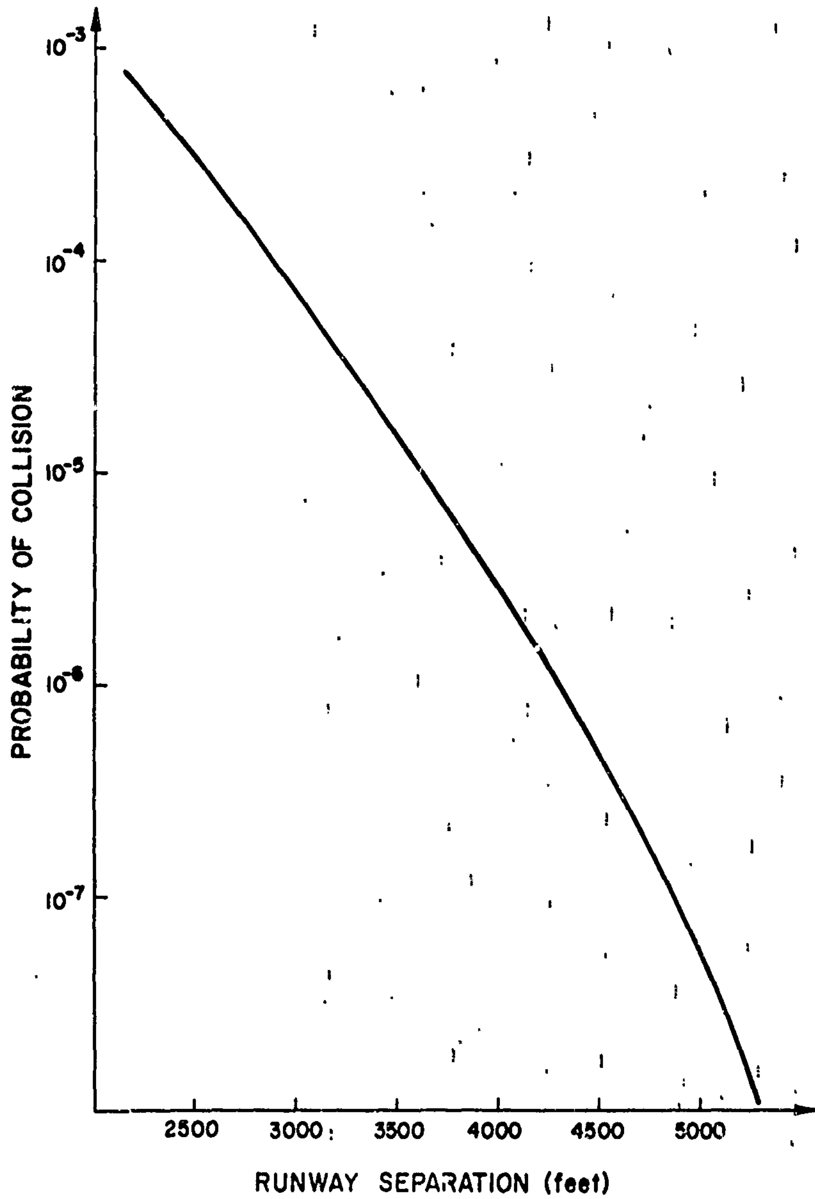

Figure 8. Probability of Collision Vs. Runway Separation.

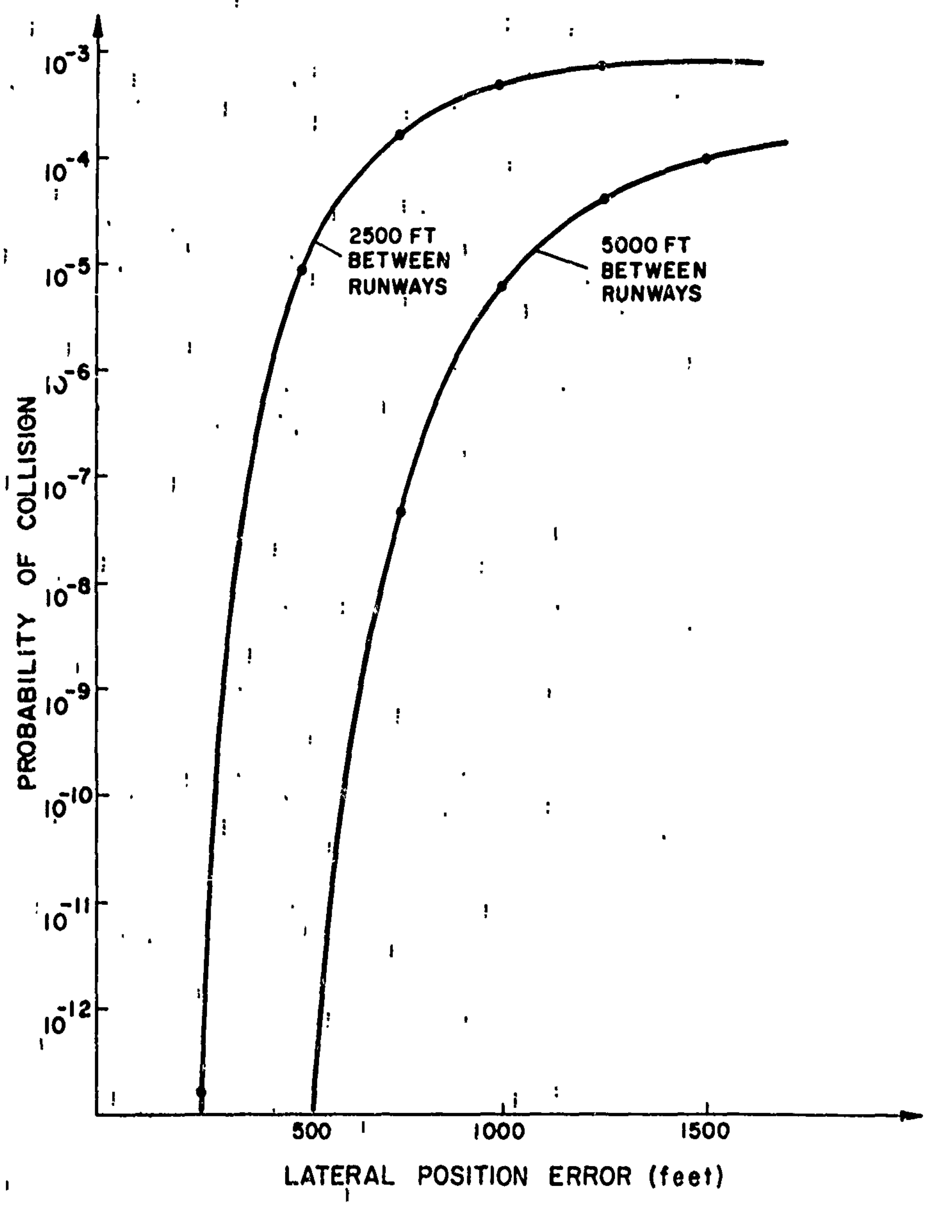

Figure 9. Probability of Collision Vs. Lateral Position Error

This result is reassuring or alarming, depending on one's view of the safety of the present 5000-ft separation and one's belief in the applicability of Gaussian statistics to the problem.

Fig. 9: (Probability of collision vs lateral navigation error.) The lateral, or y, component of position accuracy was assumed to be the same for both aircraft and was chosen to be 750 ft for the runs of Figs. 7 and 8. In the present case, however, it is allowed to vary. Fig. 9 shows that the collision risk drops drastically when lateral placement errors go below 750 or 500 ft. This suggests a trade-off between the cost of lateral guidance equipment and the cost of expanded airports.

CONCLUSION

The formulas derived allow the calculation of collision probability between two aircraft flying arbitrary curvilinear paths. They represent a generalization of the statistical-probabilistic method of analysis, which heretofore has been available only for constant altitude, rectilinear paths. The analysis results in a general integral expression for the collision probability; simpler integral formulas are derived for the cases of large relative velocity, zero relative velocity, and a spherical collision surface. In only the simplest Gaussian cases is an explicit formula available. Because high accuracy is not essential it is possible to compute by machine the appropriate integral expression formula (2) for practical cases such as curved landing and takeoff paths.

The computer calculation of formula (2) for curved approaches to parallel runways shows how the collision probability varies with time between touchdowns, runway separation, and lateral position error.

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